Sec 6.2 6.4 6.5 TWO-DIMENSIONAL AUTONOMOUS SYSTEMS

does not depend on t

A two-dimensional autonomous system is a differential equation of the form

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

where $f(x,y)$ and $g(x,y)$ are functions in two variables (see examples below).

1. $\begin{cases} x' = y \\ y' = -\sin(x) \end{cases}$

2. $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$, where $a,\ b,\ c,\ d$ are constants.

3. $\begin{cases} x' = 2xy + 4y \\ y' = x^2 + 4x + 8y \end{cases}$

4. $\begin{cases} x' = 2xy - 2x^3 \\ y' = x^2 + y^2 - 6 \end{cases}$

Equilibrium Solution (Critical Point, or Stationary Point)

An equilibrium solution of a two-dimensional autonomous system is a constant solution of the system, i.e. $(x(t), y(t)) = (a,b)$ for all t. Notice that $(x(t), y(t)) = (a,b)$ is an equilibrium solution if and only if

$$\begin{cases} 0 = x'(t) = f(a,b) \\ 0 = y'(t) = g(a,b) \end{cases}$$

constant

$Y_e = (x_e, y_e)$

The point (a,b) is usually called **equilibrium point**.

How to find equilibrium points?

We need to find the points (x,y) in the plane that satisfy **simultaneously** the equations

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

Ex

$$y' = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\begin{cases} x' = x + 2y = 0 \\ y' = 2x + 5y = 0 \end{cases}$$

Trivial Sol$\underline{n}$

($A\mathbf{y} = \vec{0}$ has the trivial sol$\underline{n}$ if $|A| \neq 0$.

$$y' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} y$$

$$\begin{cases} x' = x + 2y = 0 \\ y' = 2x - 5y = 0 \end{cases}$$ same line

Infinitly many sol$\underline{n}$

($A\mathbf{y} = \vec{0}$ has infinitly many sol$\underline{n}$ if $|A| = 0$

Example. Find the equilibrium solutions for the following equations

i. $\begin{cases} x' = 2xy + 4y = y(2x+4) = 0 \\ y' = x^2 + 4x + 8y = 0 \end{cases}$

①

$x' = y(2x+4) = 0$	$y' = x^2+4x+8y = 0$	$Y_e = (x_e, y_e)$
$y=0$	$y' = x^2+4x = 0$ $x(x+4)=0$ $x=0$ plug $y=0$	$(0,0)$ $(-4,0)$
$2x+4=0$ $\Rightarrow x=-2$	$y' = x^3+4x+8y=0$ $=(4)-8+8y=0$ $8y=4$ $y=1/2$ plug $x=-2$	$(-2, 1/2)$

3 equilibrium points

ii. $\begin{cases} x' = 2xy - 2x^3 = 2x(y-x^2) = 0 \\ y' = x^2 + y^2 - 6 = 0 \end{cases}$

$x' = 2x(y-x^2) = 0$	$y' = x^2+y^2-6=0$	$Y_e = (x_c, y_c)$
$x=0$	$y' = 0 + y^2 - 6 = 0$ $y = \pm\sqrt{6}$	$(0, +\sqrt{6})$ $(0, -\sqrt{6})$
$y = x^2$	$y' = x^2 + (x^4) - 6 = 0$ $x^4 + x^2 - 6 = 0$	

$(y^2+y-6)=0$

$(y+3)(y-2)=0$

$y=-3 \Rightarrow x^2 = -3$ (not possible)

$y=2 \Rightarrow x=2 \Rightarrow x = \pm\sqrt{2}$

$y = x^2 = 2$

DNF

"pretty much the same as a Jacobian Matrix"

Linearization at Equilibrium Points.

Preliminares:
From MATH 1205 we know that if the function $f(x)$ is differentiable at the point $x = a$, then

$$f(\mathrm{x}) \approx f(a) + f'(a)(\mathrm{x} - a)$$

whenever x sufficiently close to the point a.

In MATH 2224 we extend the preceding idea. More precisely, given a function $f(x, y)$ in two variables and a point (a, b) in the domain of f, then

$$f(\mathrm{x}, \mathrm{y}) \approx f(a, b) + f_x(a, b)(\mathrm{x} - a) + f_y(a, b)(\mathrm{y} - b)$$

whenever (x, y) is sufficiently close to (a, b). Here:

- $f_x(a, b)$ is the partial derivative of f respect to the variable x at the point (a, b).
- $f_y(a, b)$ is the partial derivative of f respect to the variable y at the point (a, b).

Now, let (a, b) be an equilibrium point of the two-dimensional autonomous system

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

then $f(a, b) = 0$, $g(a, b) = 0$ and

$$\begin{cases} x' = f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ y' = g(x, y) \approx g_x(a, b)(x - a) + g_y(a, b)(y - b) \end{cases}$$

More precisely

$$\begin{cases} x'(t) \approx f_x(a, b)(x(t) - a) + f_y(a, b)(y(t) - b) \\ y'(t) \approx g_x(a, b)(x(t) - a) + g_y(a, b)(y(t) - b) \end{cases}$$

Setting $z_1(t) = x(t) - a$ and $z_2(t) = y(t) - b$, we have $z_1'(t) = x'(t)$ and $z_2'(t) = y'(t)$. This yields,

$$\begin{cases} z_1'(t) \approx f_x(a, b)z_1(t) + f_y(a, b)z_2(t) \\ z_2'(t) \approx g_x(a, b)z_2(t) + g_y(a, b)z_2(t) \end{cases}$$

i.e.

$$\begin{cases} z_1' \approx f_x(a, b)z_1 + f_y(a, b)z_2 \\ z_2' \approx g_x(a, b)z_2 + g_y(a, b)z_2 \end{cases}$$

This motivates the following definition.

DEF. Let (a, b) be an equilibrium point of the nonlinear system of the form

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

The system has
1. only one sol$\underline{n}$ if $|A| \neq 0$
2. infinitely many if $|A| = 0$

Then, we say that the first order linear system

$$\begin{cases} z_1' = f_x(a, b) z_1 + f_y(a, b) z_2 \\ z_2' = g_x(a, b) z_2 + g_y(a, b) z_2 \end{cases}$$

is the linearized system (or linearization) at the equilibrium point (a, b).

Example: Develop the linearized-system approximation for each of the equilibrium points of the nonlinear autonomous system

$$\begin{cases} x' = 2xy + 4y & = f \\ y' = x^2 + 4x + 8y & = g \end{cases}$$

Also, determine the stability characteristics of the linearized system in each case.

1. Equilibrium pts: $(0, 0), (-4, 0), (-2, \frac{1}{2})$

2. Jacobian Matrix

$f = 2xy + 4y$

$g = x^2 + 4x + 8y$

$$J(x,y) = \begin{bmatrix} \overset{f_x}{2y} & \overset{f_y}{2x+4} \\ \underset{g_x}{2x+4} & \underset{g_y}{8} \end{bmatrix}$$

$$\mathbb{Z}' = \begin{bmatrix} 2y & 2x+4 \\ 2x+4 & 8 \end{bmatrix} \mathbb{Z} \qquad \mathbb{Z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

3. check stability of eq. points in nonlinear
$= $ stability of $(0, 0)$ in linear $\mathbb{Z}' = J\mathbb{Z}$

(extra page)

④

$(0, 0)$

$$\mathbb{Z}' = \begin{bmatrix} 0 & 4 \\ 4 & 8 \end{bmatrix}_{\substack{x=0 \\ y=0}} \mathbb{Z}$$

Eigen values of J:

$$(J - \lambda I) = \begin{vmatrix} -\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix}$$

$$= -\lambda(8-\lambda) - 16$$

$$= \lambda^2 - 8\lambda - 16 = 0$$

$$\lambda = 4 \pm \sqrt{32}$$

$$= 4 \pm 4\sqrt{2}$$

$\lambda = \oplus \ominus$

Stability of $(0,0)$: unstable (saddle)

Stability of $(0,0)$ in non linear system

$(-4, 0)$

$$\mathbb{Z}' = \begin{bmatrix} 0 & -4 \\ -4 & 8 \end{bmatrix}_{\substack{x=-4 \\ y=0}} \mathbb{Z}$$

$$|J - \lambda I| = \begin{bmatrix} -\lambda & -4 \\ -4 & 8-\lambda \end{bmatrix}$$

$$= \lambda^2 - 8\lambda - 16 = 0$$

$$\lambda = 4 \pm 4\sqrt{2}$$

$\oplus$ $\ominus$

Stability $(0,0)$ unstable (saddle)

= stability of $(-4, 0)$

$(-2, \frac{1}{2})$

$$\mathbb{Z}' = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}_{\substack{x=-2 \\ y=\frac{1}{2}}} \mathbb{Z}$$

$\lambda = 1, 8$

$\oplus$ $\oplus$

unstable (node)

= stability of $(-2, \frac{1}{2})$

One ODE | System | Polar ODE | Polar system

dx/dt = 2x*y+4y
dy/dt = x^2 + 4x +8y

Variables: y vs. x

-5 ≤ *x* ≤ 4 with 20 segments
-3 ≤ *y* ≤ 3 with 15 segments

▸ Plot numerical solution curves

▸ Plot other functions

▸ Presets

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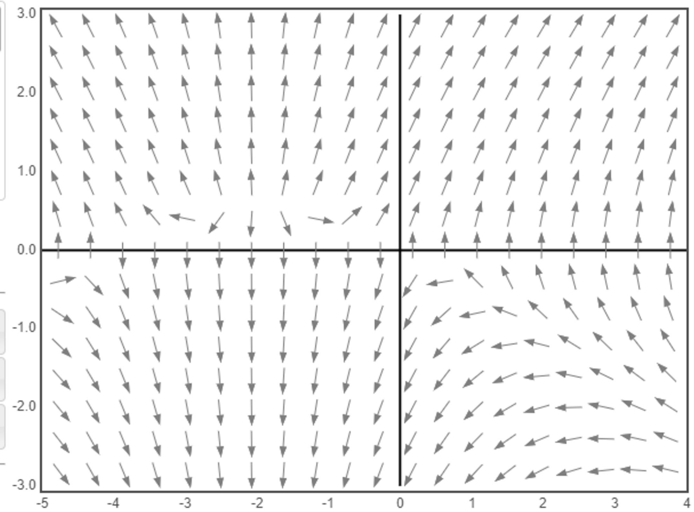

One ODE | System | Polar ODE | Polar system

dx/dt = 4y
dy/dt = 4x+8y

Variables: y vs. x

-4 ≤ *x* ≤ 4 with 20 segments
-3 ≤ *y* ≤ 3 with 15 segments

▸ Plot numerical solution curves

▸ Plot other functions

▸ Presets

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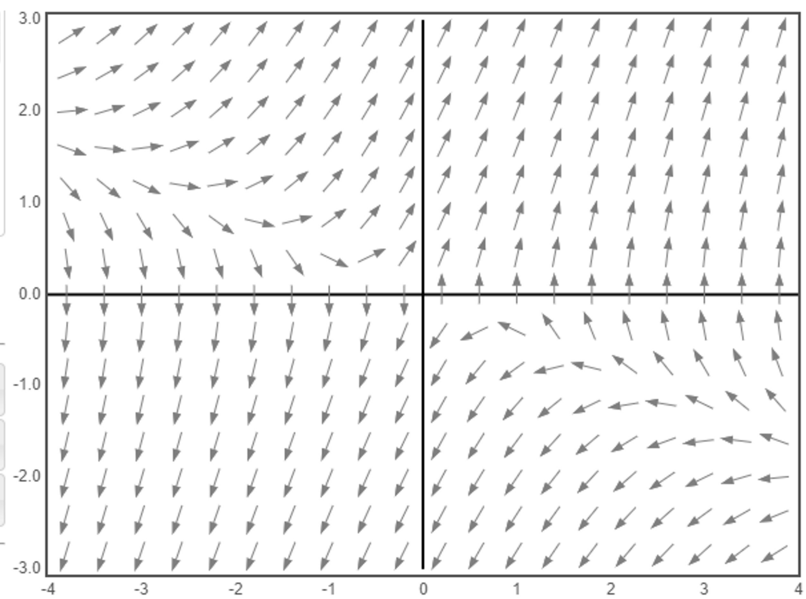

One ODE | System | Polar ODE | Polar system

$dx/dt =$ x
$dy/dt =$ 8y

Variables: y vs. x

-4 ≤ x ≤ 4 with 20 segments
-3 ≤ y ≤ 3 with 15 segments

▸ Plot numerical solution curves

▸ Plot other functions

▸ Presets

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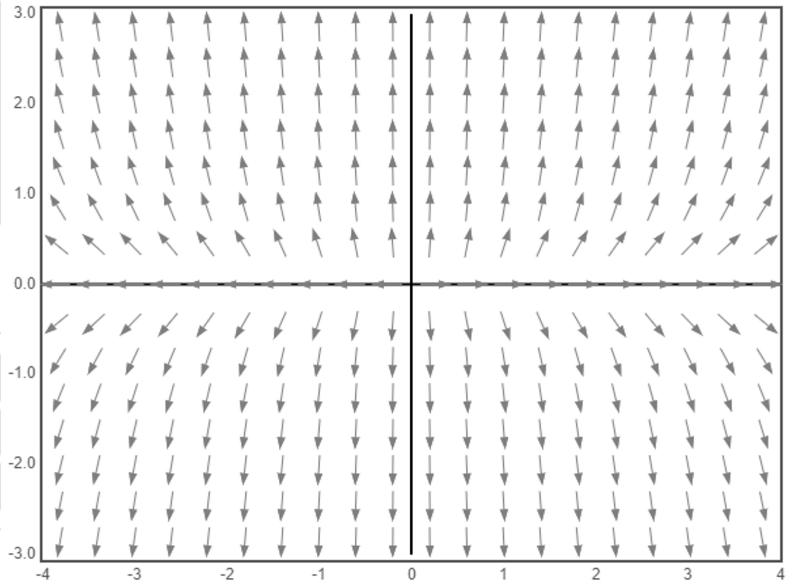

One ODE | System | Polar ODE | Polar system

$dx/dt =$ -4y
$dy/dt =$ -4x+8y

Variables: y vs. x

-4 ≤ x ≤ 4 with 20 segments
-3 ≤ y ≤ 3 with 15 segments

▸ Plot numerical solution curves

▸ Plot other functions

▸ Presets

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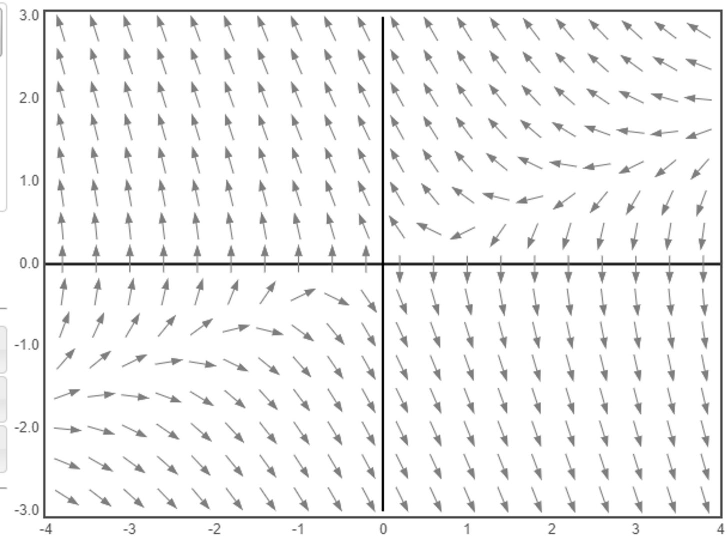

Classification of equilibrium points for nonlinear autonomous systems.

(theorem 6.4 textbook)

Theorem Let

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

be a **nonlinear** autonomous system. Suppose that (x_0, y_0) is an equilibrium point and let A be the matrix that represents the linearization at the equilibrium point (x_0, y_0); that is,

$$A = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}$$

If the matrix A is invertible, we have the following classification according to the eigenvalues of A.

- Real eigenvalues and both negative: (x_0, y_0) behaves like an **asymptotically stable node**.
- Real eigenvalues and both positive: (x_0, y_0) behaves like an **unstable node**.
- Real eigenvalues of opposite sign: (x_0, y_0) behaves like a **saddle point**.
- Complex eigenvalues, $a \pm bi$ with $a < 0$: (x_0, y_0) behaves like an **asymptotically stable focus**.
- Complex eigenvalues, $a \pm bi$ with $a > 0$: (x_0, y_0) behaves like an **unstable focus**.
- Complex eigenvalues, $a \pm bi$ with $a = 0$: **no conclusions can be drawn** about the stability properties of the equilibrium point (x_0, y_0).

Ex1. Consider the nonlinear autonomous system

$$\begin{cases} x' = 2xy - 2x^3 = f(xy) \\ y' = x^2 + y^2 - 6 = g(xy) \end{cases}$$

Perform a stability analysis at the equilibrium point $(-\sqrt{2}, 2)$.

① linearize

$$J = \begin{bmatrix} \overset{f_x}{2y - 6x^2} & \overset{f_y}{2x} \\ \underset{g_x}{2x} & \underset{g_y}{2y} \end{bmatrix} \text{ Evaluate at } \begin{matrix} x = -\sqrt{2} \\ y = 2 \end{matrix} = \begin{bmatrix} -8 & -2\sqrt{2} \\ -2\sqrt{2} & 4 \end{bmatrix}$$

$$z' = \begin{bmatrix} -8 & -2\sqrt{2} \\ -2\sqrt{2} & 4 \end{bmatrix}$$

② stability of $(-\sqrt{2}, 2) \simeq$ stability of $(0,0)$ in $z' = Jz$

$$|J - I\lambda| = \begin{vmatrix} -8-\lambda & -2\sqrt{2} \\ -2\sqrt{2} & 4-\lambda \end{vmatrix} = (-8-\lambda)(4-\lambda) - 8$$

$$= \lambda^2 + 4\lambda - 40 = 0$$

$\lambda = -2 \pm \sqrt{44}$ ⊕ ⊖ ← real values

unstable

saddle point